A thicc derivation of Maxwell's wave equations

Corey Anderson

Friday 25th August, 2023

Contents

1 Identities

To derive Maxwell's wave equations, we use the following identity:

$$\boxed{\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{V} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{V}) - \boldsymbol{\nabla}^2 \mathbf{V}} \tag{1}$$

the proof of this identity has been left as an exercise for the reader. We will also make use of Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2}$$

$$\boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \tag{4}$$

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{5}$$

2 The Electric field

We start by deriving Gauss's law using the electric field:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 E = -\boldsymbol{\nabla} \times \frac{\partial \mathbf{B}}{\partial t} \tag{6}$$

$$\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \boldsymbol{\nabla}^2 E = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) \tag{7}$$

$$\frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho - \boldsymbol{\nabla}^2 E = -\frac{\partial}{\partial t}\left(\mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \tag{8}$$

$$\boldsymbol{\nabla}^2 E - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho + \mu_0 \frac{\partial}{\partial t}\mathbf{J} \tag{9}$$

$$\tag{10}$$

We now define the d'Alembert operator as

$$\Box^2 = \boldsymbol{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}. \tag{11}$$

So

$$\boxed{\Box^2 \mathbf{E} = \frac{1}{\epsilon_0}\boldsymbol{\nabla}\rho + \mu_0 \frac{\partial}{\partial t}\mathbf{J}} \tag{12}$$

which is the Maxwell's wave equation for electric fields. For an electromagnetic wave propagating through a vacuum with no charges, we have zero charge density and current density. Therefore we have

$$\Box^2 E = 0 \tag{13}$$

3 The Magnetic Field

By a similar process:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{B} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \boldsymbol{\nabla}^2 \mathbf{B} = \mu_0 \boldsymbol{\nabla} \times \mathbf{J} + \frac{1}{c^2} \boldsymbol{\nabla} \times \frac{\partial \mathbf{E}}{\partial t} \tag{14}$$

$$-\boldsymbol{\nabla}^2 B = \mu_0 \boldsymbol{\nabla} \times \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} (\boldsymbol{\nabla} \times \mathbf{E}) \tag{15}$$

$$= \mu_0 \boldsymbol{\nabla} \times \mathbf{J} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} \tag{16}$$

$$\boldsymbol{\nabla}^2 B - \frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J} \tag{17}$$

Therefore we have our wave equation for magnetism:

$$\square^2 \mathbf{B} = -\mu_0 \boldsymbol{\nabla} \times \mathbf{J}. \tag{18}$$

For a magnetic field propagating in a vacuum, we have

$$\square^2 \mathbf{B} = 0. \tag{19}$$

The combination of Equation 19 and 13 describe an electromagnetic wave propagating through a vacuum....*mic drop*