Gauge Transformations

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In this paper we will investigate the potential formulation of Maxwell's equations which will naturally lead us to the concept of gauge transformations. Maxwell's equations can be written in the differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{Gauss's Law} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{Monopole Law} \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{Faraday's Law} \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{Ampere's Law} \tag{4}$$

To write these equations in terms of potentials, we note that $\mathbf{B}$ is divergentless. Thus,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

and inserting this into Faraday's law gives

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \tag{6}$$

$$= -\nabla \times \frac{\partial \mathbf{A}}{\partial t} \tag{7}$$

so

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0. \tag{8}$$

Since the curl is zero, we can then introduce the potential term

$$\mathbf{E} - \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \tag{9}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \tag{10}$$

Equation 10 is the electric field in terms of the scalar and vector potential. Note that it reduces to the familiar case $\mathbf{E} = -\nabla V$ for time independent potentials. We also require that Equation 10 satisfy Gauss's Law, so substituting gives

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon_0} \tag{11}$$

which is simply the Poisson equation in the static case. Putting Equations 5 and 10 into Ampere's Law gives

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \nabla \left(\frac{\partial V}{\partial t} \right) - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2}. \tag{12}$$

Now using the vector identity $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\times\mathbf{A} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A}) - \nabla^2\mathbf{A}$, and rearranging the terms gives

$$\left(\nabla^2\mathbf{A} - \mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\right) - \boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{A} + \mu_0\epsilon_0\frac{\partial V}{\partial t}\right) = -\mu_0\mathbf{J}. \tag{13}$$

Equations 11 and 13 gives the equivalent information as Maxwell's equations. It is important to note that Equations 5 and 10 do not uniquely define potentials. We are free to play around with V and $\mathbf{A}$ as long as nothing happens to $\mathbf{E}$ and $\mathbf{B}$.
Suppose the fields are left invariant under some transformation of the potentials. The transformations can be written as

$$V \to V + \beta \qquad \text{and} \qquad \mathbf{A} \to \mathbf{A} + \boldsymbol{\alpha}. \tag{14}$$

To leave the magnetic field invariant, we have

$$\boldsymbol{\nabla}\times\boldsymbol{\alpha} = 0, \tag{15}$$

which means α is the gradient of some scalar λ:

$$\boldsymbol{\alpha} = \boldsymbol{\nabla}\lambda. \tag{16}$$

To leave the electric field invariant, we have

$$\boldsymbol{\nabla}\beta + \frac{\partial\boldsymbol{\alpha}}{\partial t} = \mathbf{0} \tag{17}$$

$$\boldsymbol{\nabla}\left(\beta + \frac{\partial\lambda}{\partial t}\right) = \mathbf{0}. \tag{18}$$

The term in the parenthesis is independent of position, but might be dependent on time,

$$\beta = -\frac{\partial\lambda}{\partial t} + k(t). \tag{19}$$

We can absorb the $k(t)$ into λ by transforming $\lambda \to \lambda + \int_0^t k(\tau)\,d\tau$. Note that this does not change Equation 16. We have

$$V \to V - \frac{\partial\lambda}{\partial t} \tag{20}$$

$$\mathbf{A} \to \mathbf{A} + \boldsymbol{\nabla}\lambda \tag{21}$$

These are known as the gauge transformations and when you transform the scalar and vector potential in this way simultaneously, it leaves the fields unchanged. This allows us to simplify equations by choosing the appropriate gauge. While there are a variety of gauge choices, I will introduce you to the two most popular gauges.

The **Coulomb Gauge** is used in magnetostatics, with the choice

$$\boldsymbol{\nabla}\cdot\mathbf{A} = 0. \tag{22}$$

Equation 11 becomes

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}, \tag{23}$$

which is just the Poisson equation whose solution is given by

$$V = \frac{1}{4\pi\epsilon_0}\iiint \frac{\rho(\mathbf{r}',t)}{r}\,d\tau'\,. \tag{24}$$

The Coulomb Gauge is not as useful in magnetodynamics since Equation 13 is still a monstrosity after shifting to the Coulomb Gauge.

The **Lorenz Gauge** is more useful in magnetodynamics. It is given by

$$\boldsymbol{\nabla} \cdot \mathbf{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}, \tag{25}$$

and is used to eliminate the middle term in Equation 13,

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}. \tag{26}$$

Meanwhile, the scalar potential equation (Equation 13) in the Lorenz Gauge becomes

$$\nabla^2 V - \mu_0 \epsilon_0 \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}. \tag{27}$$

The Lorenz gauge treats both potentials on equal footing, that is, the same d'Alembertian operator

$$\Box^2 \equiv \nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \tag{28}$$

occurs in both Equations 26 and 27:

$$\Box^2 V = -\frac{\rho}{\epsilon_0} \tag{29}$$

$$\Box^2 \mathbf{A} = -\mu_0 \mathbf{J}. \tag{30}$$

These two equations can be regarded as the 4-dimensional version of the Poisson equation. It is clear that both potentials satisfy the inhomogeneous wave equation. The treatment of V and $\mathbf{A}$ on equal footing is useful in special relativity where the d'Alembertian is the natural generalisation of the Laplacian operator.