Differentiating under the Integral sign like an ABSOLUTE PRO

Corey Anderson

Sunday 3rd September, 2023

Consider the integral

$$\int_0^1 \frac{t^3 - 1}{\ln(t)}\, dt\,, \tag{1}$$

which we are going to evaluate by differentiating under the integral sign. First note the following differential relation

$$\frac{d}{dx} t^x = t^x \ln(t). \tag{2}$$

Proving this relation has been left as an exercise for the reader but if you get stuck you may Google or message me. This relation will come in handy later. Lets first generalise the integral such that

$$g(x) = \int_0^1 \frac{t^x - 1}{\ln(t)}\, dt\,, \tag{3}$$

and we want to evaluate $g(3)$. First take the derivative

$$\begin{aligned} g'(x) &= \frac{\partial}{\partial x} \int_0^1 \frac{t^x - 1}{\ln(t)}\, dt && (4)\\ &= \int_0^1 \frac{t^x \ln(t)}{\ln(t)}\, dt && (5)\\ &= \int_0^1 t^x\, dt && (6)\\ &= \frac{1}{x+1} t^{x+1} \bigg|_{t=0}^{t=1} && (7)\\ &= \frac{1}{x+1}. && (8) \end{aligned}$$

Integrating,

$$\begin{aligned} g(x) &= \ln|x+1| + g(0) && (9)\\ &= \ln|x+1| && (10) \end{aligned}$$

where the constant of integration is zero using Equation 3. Now we can finally find the value of the integral in Equation 1 which is

$$g(3) = \int_0^1 \frac{t^3 - 1}{\ln(t)}\, dt = \ln(4) \tag{11}$$