Fourier Analysis

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Monday 15$^{\text{th}}$ January, 2024

1 Fourier Series

1.1 Real Fourier Series

Fourier series allow us to represent periodic functions which are not everywhere continuous and differentiable. Unlike the Taylor series, functions are written in terms of cosine and sine terms. The Fourier series expansion is conventionally written as

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi kx}{T}\right) + b_k \sin\left(\frac{2\pi kx}{T}\right) \right] \tag{1}$$

where a_0, a_k and b_k are called the Fourier coefficients. To find these coefficients, we will use Fourier's trick which takes advantage of the fact the cosine and sine are orthogonal functions. We will start by multiplying Equation 1 by $\cos(2\pi px/T)$ and integrating over a full period in x:

$$\int_{x_0}^{x_0+T} f(x)\cos\left(\frac{2\pi px}{T}\right) dx = \frac{a_0}{2}\int_{x_0}^{x_0+T} \cos\left(\frac{2\pi px}{T}\right) dx \tag{2}$$

$$+ \sum_{k=1}^{\infty} a_k \int_{x_0}^{x_0+T} \cos\left(\frac{2\pi kx}{T}\right)\cos\left(\frac{2\pi px}{T}\right) dx \tag{3}$$

$$+ \sum_{k=1}^{\infty} b_k \int_{x_0}^{x_0+T} \sin\left(\frac{2\pi kx}{T}\right)\cos\left(\frac{2\pi px}{T}\right) dx \tag{4}$$

Using the orthogonality condition, and substituting in $p = 0$, Equation 2 becomes

$$\int_{x_0}^{x_0+T} f(x)\, dx = \frac{a_0}{2}\int_{x_0}^{x_0+T} dx + 0 + 0 = \frac{a_0}{2}T \tag{5}$$

which gives us the expression for the first Fourier coefficient

$$\boxed{a_0 = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\, dx} \tag{6}$$

If we instead consider $p \neq 0$, this gives us

$$\int_{x_0}^{x_0+T} f(x)\cos\left(\frac{2\pi px}{T}\right) dx = 0 + \sum_{k=1}^{\infty} a_k\delta_{kp}\int_{x_0}^{x_0+T} \cos^2\left(\frac{2\pi kx}{T}\right) dx \tag{7}$$

$$= a_k\frac{T}{2} \tag{8}$$

This gives us the second Fourier coefficient which is:

$$\boxed{a_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\cos\left(\frac{2\pi px}{T}\right) dx} \tag{9}$$

The b_k coefficient can be obtained by multiplying Equation 1 by $\sin(2\pi px/T)$ and integrating

$$\boxed{b_k = \frac{2}{T}\int_{x_0}^{x_0+T} f(x)\sin\left(\frac{2\pi px}{T}\right) dx} \tag{10}$$

1.2 Complex Fourier Series

Since the Fourier series consists of both cosine and sine terms, we can conveniently write it in terms of complex exponentials, making use of $e^{ix} = \cos x + i \sin x$. The complex Fourier series expansion can be written as

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/T} \tag{11}$$

Similar to the previous section, the Fourier coefficients can be derived by multiplying Equation 11 by $e^{2\pi ipx/T}$ before integrating and using the orthogonality relation to give

$$\boxed{c_k = \frac{1}{L}\int_{x_0}^{x_0+L} f(x)e^{-2\pi ikx/T}\,dx} \tag{12}$$

The complex Fourier coefficients have the following relationships with the real coefficients:

$$c_k = \frac{1}{2}(a_k - ib_k) \tag{13}$$

$$c_{-k} = \frac{1}{2}(a_k + ib_k) \tag{14}$$

Note that if $f(x)$ is real then $c_{-k} = c_k^*$.

1.3 Parseval's Theorem

Parseval's theorem is a useful way to relate the Fourier coefficients to the functions they describe. It states that

$$\frac{1}{T}\int_{x_0}^{x_0+T} |f(x)|^2\,dx = \sum_{k=-\infty}^{\infty} |c_k|^2 \tag{15}$$

To prove this relation, consider two functions $f(x)$ and $g(x)$, where

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx/T} \tag{16}$$

$$g(x) = \sum_{k=-\infty}^{\infty} \gamma_k e^{2\pi ikx/T} \tag{17}$$

Now,

$$f(x)g^*(x) = \sum_{k=-\infty}^{\infty} c_k g^*(x) e^{2\pi ikx/T} \tag{18}$$

Integrating Equation 18 over one period and dividing by T, we have

$$\frac{1}{T}\int_{x_0}^{x_0+L} f(x)g^*(x)\,dx = \sum_{k=-\infty}^{\infty} c_k \frac{1}{T}\int_{x_0}^{x_0+L} g^*(x)e^{2\pi ikx/T}\,dx \tag{19}$$

$$= \sum_{k=-\infty}^{\infty} c_k \left[\frac{1}{T}\int_{x_0}^{x_0+L} g(x)e^{-2\pi ikx/T}\,dx\right]^* \tag{20}$$

$$= \sum_{k=-\infty}^{\infty} c_k \gamma_k^* \tag{21}$$

Setting $f(x) = g(x)$ and noting that this implies $c_k = \gamma_k$, results in Equation 15 as required.