Integrating a BIG Gaussian boii

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The Gaussian is defined by the following function

$$f(x) = e^{-ax^2} \tag{1}$$

for some real number a. To integrate f, we will consider the case where $a = 1$ and then generalise it later. The integral to be evaluated is

$$I = \int_{-\infty}^{\infty} e^{-x^2}\, dx\,. \tag{2}$$

This integral can not be evaluated with conventional methods such as integration by parts and substitution, so we'll use another method. We begin by squaring

$$I^2 = \left[\int_{-\infty}^{\infty} e^{-x^2}\, dx\right]^2 = \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \tag{3}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx\, dy\,. \tag{4}$$

Now convert from cartesian to polar coordinates such that

$$\begin{cases} dx\, dy = r\, dr\, d\theta \\ x^2 + y^2 = r^2. \end{cases} \tag{5}$$

Equation 4 becomes

$$I^2 = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta\,. \tag{6}$$

Now this is in the form where we can use a substitution of

$$u = e^{-r^2}, \tag{7}$$

so that

$$du = -2re^{-r^2} \longrightarrow -\frac{1}{2}\, du = re^{-r^2} \tag{8}$$

and using Equation 7 to determine the new integration bounds gives us the alternate form of Equation 6 to be

$$I^2 = \int_0^{2\pi}\int_1^{0} -\frac{1}{2}\, du\, d\theta \tag{9}$$

$$= 2\pi \cdot \frac{1}{2}\int_0^1 du \tag{10}$$

$$= \pi. \tag{11}$$

Thus the Gaussian integral is given by

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{12}$$

In the next I will generalise Equation 12 to work with any real coefficient in the exponent.

We now attempt to integrate

$$I = \int_{-\infty}^{\infty} e^{-ax^2}. \tag{13}$$

First let

$$u = \sqrt{a}x, \tag{14}$$

so that

$$I = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2}\, du \tag{15}$$

$$= \sqrt{\frac{\pi}{a}} \tag{16}$$

Therefore, the generalised Gaussian integral is

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2} = \sqrt{\frac{\pi}{a}}} \tag{17}$$