Noether's Theorem
A paper on my favourite mathematician <3

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Noether's theorem is easily one of my favourite theorems in mathematics, so I have included multiple methods of derivation. Each derivation heavily relies on Lagrangian Mechanics so review my paper on that topic before proceeding if you are not familiar.

1 Noether's Theorem

Noether's Theorem states that

$$\boxed{\text{Every continuous symmetry implies a conservation law.}} \tag{1}$$

For example, time translation symmetry gives conservation of energy; space translation symmetry gives conservation of momentum; rotation symmetry gives conservation of angular momentum, and so on.

2 Derivations

2.1 Classical Mechanics of a Point Particle

Suppose the conserved quantity takes the form

$$C = p\frac{dq(s)}{ds} \tag{2}$$

and taking the time-derivative

$$\frac{dC}{dt} = \dot{p}\frac{dq(s)}{ds} + p\frac{d\dot{q}(s)}{ds} \tag{3}$$

$$= \frac{\partial \mathcal{L}}{\partial q}\frac{dq(s)}{ds} + \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{d\dot{q}(s)}{ds} \tag{4}$$

$$= \frac{d}{ds}[\mathcal{L}(q(s), \dot{q}(s), t)]. \tag{5}$$

This means that if the Lagrangian has a symmetry classified by the transformation parameter s,

$$\frac{d}{ds}[\mathcal{L}(q(s), \dot{q}(s), t)] = 0. \tag{6}$$

Then

$$\frac{dC}{dt} = 0 \tag{7}$$

and so the quantity C is conserved.

2.2 Classical Field Theory

Consider a transformation to a field ϕ given by

$$\phi \rightarrow \phi e^{-i\alpha} \tag{8}$$

Lets impose the condition that the Lagrangian is invariant under this transformation, this gives

$$0 = \frac{\delta S}{\delta \alpha} = \int d^4x \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_n} \frac{d\phi_n}{d\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{d(\partial_\mu \phi_n)}{d\alpha} \right] \tag{9}$$

$$= \int d^4x \sum_n \left[\frac{\partial \mathcal{L}}{\partial \phi_n} \frac{d\phi_n}{d\alpha} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) \frac{d\phi_n}{d\alpha} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{d\phi_n}{d\alpha} \right) \right] \tag{10}$$

But we must also impose the condition of least action, which means the Euler-Lagrange equation must be satisfied as well:

$$\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \right) = 0, \tag{11}$$

and Equation 10 becomes

$$0 = \int d^4x \sum_n \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{d\phi_n}{d\alpha} \right) \right] \tag{12}$$

which means that

$$\sum_n \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{d\phi_n}{d\alpha} \right) = 0. \tag{13}$$

Now let the 4-current density be defined as

$$J^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \frac{d\phi_n}{d\alpha} \tag{14}$$

so that

$$\partial_\mu J^\mu = 0. \tag{15}$$

Note that this is a divergence which means that there is no point source of current pouring things in. If the 4-current density depends on some charge density ρ and a current density $\mathbf{J}$, so that $J^\mu = \begin{pmatrix} \rho & \mathbf{J} \end{pmatrix}^T$, then

$$0 = \partial_\mu J^\mu = \begin{pmatrix} \frac{\partial}{\partial t} & -\boldsymbol{\nabla} \end{pmatrix} \cdot \begin{pmatrix} \rho \\ \mathbf{J} \end{pmatrix} \tag{16}$$

so

$$\boxed{\frac{\partial \rho}{\partial t} - \boldsymbol{\nabla} \cdot \mathbf{J} = 0} \tag{17}$$

which is the familiar continuity equation. To see the physical interpretation of this, we integrate this within a volume V whose boundary is ∂V, and applying the divergence theorem and rearranging, we get

$$\frac{d}{dt} \iiint_V \rho \, dV = \iint_{\partial V} \mathbf{j} \cdot \mathbf{dS}. \tag{18}$$

This equation states that the rate of change of the charge within volume V is equal to the current flux entering through the boundary ∂V. This is essentially just the conservation of charge which makes sense intuitively.