Eigenvalues of the Mobius Strip - Fact checking Tony Stark in Avengers

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Saturday 26th August, 2023

1 Boundary Conditions

Consider a rectangular strip of height w and length l that has been inverted into a mobius strip. The electron is confined on the mobius strip and changes face for every loop. The boundary conditions are therefore:

$$\psi(x, -\frac{w}{2}) = \psi(x, \frac{w}{2}) = 0 \tag{1}$$

$$\psi(x + l, y) = \psi(x, -y) \tag{2}$$

2 The Eigensystem

The Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x, y) + V(x, y)\psi(x, y) = E\psi(x, y). \tag{3}$$

Since the electron is confined to the mobius strip of zero potential:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x, y) = E\psi(x, y). \tag{4}$$

It can be assumed that the wavefunction can be split like $\psi(x, y) = f(x)g(y)$, thus:

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f(x)g(y) = Ef(x)g(y) \tag{5}$$

$$g(y)\frac{\partial^2 f}{\partial x^2} + f(x)\frac{\partial^2 g}{\partial y^2} = -\frac{2mE}{\hbar^2}f(x)g(y) \tag{6}$$

$$\frac{1}{g(y)}\frac{\partial^2 f}{\partial x^2} + \frac{1}{f(x)}\frac{\partial^2 g}{\partial y^2} = -\frac{2m}{\hbar^2}(E_x + E_y) \tag{7}$$

$$\frac{\hbar^2}{2m}(k_x^2 + k_y^2) = E_x + E_y, \tag{8}$$

where we have defined k_x and k_y as the second partial derivative terms. Using this, we have the following differential equations:

$$\frac{1}{f(x)}\frac{\partial^2 f}{\partial x^2} = -k_x^2 \quad \text{and} \quad \frac{1}{g(y)}\frac{\partial^2 g}{\partial y^2} = -k_y^2. \tag{9}$$

It is trivial to find the solutions to the two equations as:

$$f(x) = c_1\cos(k_x x) + c_2\sin(k_x x) \tag{10}$$

$$g(y) = c_3\cos(k_y y) + c_4\sin(k_y y). \tag{11}$$

Now we are in the position to use our boundary conditions for the mobius strip. Using the first boundary condition:

$$c_3 \cos\Big(k_y \frac{w}{2}\Big) + c_4 \sin\Big(k_y \frac{w}{2}\Big) = c_3 \cos\Big(k_y \frac{w}{2}\Big) - c_4 \sin\Big(k_y \frac{w}{2}\Big) \tag{12}$$

$$2c_4 \sin\Big(k_y \frac{w}{2}\Big) = 0. \tag{13}$$

Equation 13 presents two possibilities, either $c_4 = 0$ or $\sin(k_y w/2) = 0$

Case 1: $c_4 = 0$, which leads to

$$g(y) = c_3 \cos(k_y y). \tag{14}$$

The wavefunction is zero at the boundary, so:

$$g\Big(\frac{w}{2}\Big) = c_3 \cos\Big(k_y \frac{w}{2}\Big) = 0 \tag{15}$$

Hence,

$$k_y \frac{w}{2} = \frac{(2n+1)}{2}\pi \;\longrightarrow\; k_y = \frac{2n+1}{w}\pi \tag{16}$$

Case 2: $\sin(k_y w/2) = 0$, which leads to

$$k_y \frac{w}{2} = n\pi \;\longrightarrow\; k_y = \frac{2\pi n}{w}, \tag{17}$$

and

$$g\Big(\frac{w}{2}\Big) = c_3 \cos\Big(\frac{2\pi n}{w}\frac{w}{2}\Big) = c_3 \cos(\pi n) = 0, \tag{18}$$

so $c_3 = 0$ and

$$g(y) = c_4 \sin(k_y y) \tag{19}$$

We have derived the two cases for our first boundary condition. To summarise:

$$\begin{cases} \textbf{Case 1:}\ c_4 = 0,\ k_y = \frac{2n+1}{w}\pi,\ g(y) = c_3 \cos(k_y y) \\ \textbf{Case 2:}\ c_3 = 0,\ k_y = \frac{2\pi n}{w},\ g(y) = c_4 \sin(k_y y) \end{cases} \tag{20}$$

Now lets define new constants:

$$A_1 = c_1 c_3,\ A_2 = c_2 c_3,\ A_3 = c_1 c_4,\ A_4 = c_2 c_4. \tag{21}$$

We require that our wavefunction satisfies boundary condition two. Starting from the cases,

Case 1:

$$\psi(x, y) = A_1 \cos(k_x x) \cos(k_y y) + A_2 \sin(k_x x) \cos(k_y y) \tag{22}$$

Using the second boundary condition:

$$A_1 \cos(k_x[x + l]) \cos(k_y y) + A_2 \sin(k_x[x + l]) \cos(k_y y) = A_1 \cos(k_x x) \cos(k_y y) + A_2 \sin(k_x x) \cos(k_y y) \tag{23}$$

$$A_1 \cos(k_x[x + l]) + A_2 \sin(k_x[x + l]) = A_1 \cos(k_x x) + A_2 \sin(k_x x) \tag{24}$$

This equation needs to hold for all x so it needs to hold for $x = 0$. Therefore:

$$A_1 \cos(k_x l) + A_2 \sin(k_x l) = A_1 \tag{25}$$

so

$$k_x l = 2\pi m \ \longrightarrow \ k_x = \frac{2\pi m}{l} \tag{26}$$

Case 2:
Through a similar process, it is easy to show that

$$k_x = \frac{2m+1}{l}\pi \tag{27}$$

Now we have two k_x expressions and two k_y expressions corresponding to the two cases. Thus, using Equation 8:

$$E_{mn} = \frac{\hbar^2}{2m}\begin{cases} (\frac{2\pi m}{l})^2 + (\frac{2n+1}{w}\pi)^2 & \text{- Case 1} \\ (\frac{2m+1}{l}\pi)^2 + (\frac{2\pi n}{w})^2 & \text{- Case 2} \end{cases} \tag{28}$$

Equation 28 is the energy eigenvalues of the $\psi_{mn}(x, y)$ eigenstate for a particle under mobius strip boundary conditions. Despite what Tony Stark may have lead you to believe, finding the eigenvalues of the mobius strip does not enable time travel, but it is an interesting problem nonetheless.