The Gamma Function

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The two basic properties the Gamma Function Γ are:

$$\Gamma(x+1) = x\Gamma(x) \quad \text{for all } x > 0. \tag{1}$$
$$\Gamma(n+1) = n! \quad \text{where } n \in \mathbb{N} \tag{2}$$

Consider the integral of an elementary function $f(x) = e^{-x}$:

$$\int_0^\infty e^{-ax}\,dx = \frac{1}{a}. \tag{3}$$

Differentiating under the integral sign with respect to a, we get

$$\int_0^\infty xe^{-ax}\,dx = \frac{1}{a^2}, \qquad \int_0^\infty x^2e^{-ax}\,dx = \frac{2}{a^3}, \qquad \int_0^\infty x^3e^{-ax}\,dx = \frac{6}{a^4}. \tag{4}$$

Generalising this sequence,

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}} \tag{5}$$

Setting $a = 1$, we have

$$n! = \int_0^\infty x^n e^{-x}\,dx \tag{6}$$

Now according to Equation 2, the Gamma function must be defined by

$$\boxed{\Gamma(n) = \int_0^\infty x^{n-1}e^{-ax}\,dx} \tag{7}$$

This equation is significant as it allows us to extend the computation of factorials to $\mathbb{R}$ instead of being confined only in $\mathbb{Z}$. Figure 1 is a plot of the Gamma function. It is undefined to negative integers and defined everywhere else. However, note that factorials are only properly defined for positive integers (see Equation 2). The non-integer cases are merely interpolations of the positive integer dataset. There are many ways to interpolate this dataset, but the Gamma function happens to be the most useful.

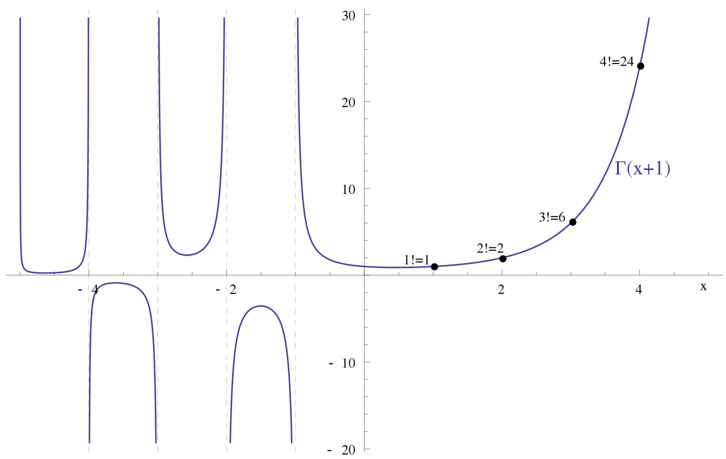

Figure 1: $\Gamma(x+1)$ interpolates the factorial function to non-integer values

By now, you may be wondering why the Gamma function isn't defined as

$$\Gamma(n) = \int_0^\infty x^n e^{-ax}\,dx\,. \tag{8}$$

The answer to this question can be found in Riemann's Zeta Function, By H. M. Edwards, footnote on page 8: *"Legendre's reasons for considering* $(n-1)!$ *instead of* $n!$ *are obscure but, whatever the reason, this notation prevailed in France and, by the end of the nineteenth century, in the rest of the world as well".*