The Virial Theorem

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Saturday 7th October, 2023

The virial theorem is a fundamental concept in classical mechanics that relates the average values of kinetic and potential energies in a system of interacting particles. This theorem provides insights into the equilibrium and stability of such systems, shedding light on their dynamical and physical properties. Originally formulated in the context of celestial mechanics, the virial theorem has found wide applications in various areas of physics, including astrophysics, quantum mechanics, and statistical mechanics.

1 Derivation

Consider a general system of mass points with position vectors $\mathbf{r_i}$ and applied forces $\mathbf{F_i}$. Consider the scalar product G,

$$G \equiv \sum_i \mathbf{p_i} \cdot \mathbf{r_i} \tag{1}$$

where i sums over all particles in the system. The time derivative of G is

$$\begin{align} \frac{dG}{dt} &= \sum_i \mathbf{p_i} \cdot \dot{\mathbf{r}}_\mathbf{i} + \sum_i \dot{\mathbf{p}}_\mathbf{i} \cdot \mathbf{r_i} \tag{2} \\ &= \sum_i m\dot{\mathbf{r}}_\mathbf{i} \cdot \dot{\mathbf{r}}_\mathbf{i} + \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \tag{3} \\ &= 2T + \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \tag{4} \end{align}$$

If we now average over a period τ

$$\frac{1}{\tau}\int_0^\tau \frac{dG}{dt}\,dt = \frac{G(\tau) - G(0)}{\tau} = \langle 2T \rangle + \left\langle \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle \tag{5}$$

If the particles have finite position and velocities for all time τ, then there is an upper bound for G. This implies that choosing $\tau \to \infty$ means that $\frac{G(\tau)-G(0)}{\tau} \to 0$. So for a sufficient time frame, the LHS of Equation 5 tends to zero giving the virial theorem

$$\boxed{\langle T \rangle = -\frac{1}{2}\left\langle \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle} \tag{6}$$

For a conservative central force $\mathbf{F} = -\boldsymbol{\nabla} U$ the Virial theorem becomes

$$\langle T \rangle = \frac{1}{2}\left\langle \boldsymbol{\nabla} U \cdot \mathbf{r} \right\rangle = \frac{1}{2}\left\langle r\frac{\partial U}{\partial r} \right\rangle \tag{7}$$

If the potential is in the form $U = kr^{n+1}$, that is, $F = -k(n+1)r^n$, then by eliminating k, $r\frac{\partial U}{\partial r} = (n+1)U$. This reduces the Virial theorem down to

$$\langle T \rangle = \frac{n+1}{2}\langle U \rangle \tag{8}$$

which is extremely useful as we'll see in the examples on the next page.

2 Applications

2.1 Classical Harmonic Oscillator

The potential for a classical harmonic oscillator is $U = kx^2$, in this case $n = 1$. So from Virial's theorem, $\langle T \rangle = \langle U \rangle$ so the average kinetic and potential energies are equal and is each half the total energy. This is what we expect from classical mechanics.

2.2 The Inverse Square Law

The inverse square law has a potential of $U = \frac{k}{r}$. In this case, $n = -2$ so from Virial's theorem, $\langle T \rangle = -\frac{1}{2} \langle U \rangle$. The total energy is $\langle E \rangle = \langle T \rangle + \langle U \rangle = -\frac{1}{2} \langle U \rangle + \langle U \rangle = \frac{1}{2} \langle U \rangle$. Examples of this is Bohr's semi-quantum model of the hydrogen atom where the kinetic energy of the bound electron is half of the potential energy. The same result occurs for planetary motion in the solar system.

2.3 The Ideal Gas Law

Consider an ideal gas inside a sealed container. From the Equipartition theorem, the average kinetic energy per atom is $\frac{3}{2}kT$. Thus the total energy for a gas containing N atoms is $\frac{3}{2}NkT$. For an ideal gas, the pressure force from the walls of the container. This is given by

$$d\mathbf{F}_i = -\hat{\mathbf{n}} P \, dA \tag{9}$$

The RHS of the Virial theorem is therefore

$$-\frac{1}{2} \left\langle \sum_i \mathbf{F_i} \cdot \mathbf{r_i} \right\rangle = \frac{P}{2} \iint \hat{\mathbf{n}} \cdot \mathbf{r_i} \, dA \tag{10}$$

Using the divergence theorem,

$$\frac{P}{2} \iint \hat{\mathbf{n}} \cdot \mathbf{r_i} \, dA = \frac{P}{2} \iiint \boldsymbol{\nabla} \cdot \mathbf{r} \, dV = \frac{3P}{2} \iiint dV = \frac{3PV}{2} \tag{11}$$

Thus, from the Virial theorem (Equation 6), we have the familiar ideal gas law

$$PV = NkT \tag{12}$$

2.4 The Mass of Galaxies and Dark Matter

Assuming a spherically-symmetric cluster of N galaxies, each of mass m, then the total mass of the cluster is $M = Nm$. A crude estimate of the cluster potential energy is

$$\langle U \rangle \approx -\frac{GM^2}{R} \tag{13}$$

where R is the radius of the cluster. The average kinetic energy per galaxy is $\frac{1}{2}m \langle v \rangle^2$ where $\langle v \rangle$ is the average velocity with respect to the center of mass of the cluster. Then the total kinetic energy of the cluster becomes

$$\langle T \rangle \approx \frac{Nm \langle v \rangle^2}{2} = \frac{M \langle v \rangle^2}{2} \tag{14}$$

The gravitational force follows the inverse square law so from Section 2.2 we have

$$\langle T \rangle = -\frac{1}{2} \langle U \rangle \tag{15}$$

Thus combining Equations 13, 14 and 15, we get

$$M \approx \frac{R \langle v \rangle^2}{G} \tag{16}$$

However, this estimate is much larger than the value estimated from the luminosity of the cluster. There are two possible resolutions: There is some special matter which we have not detected contributing to the mass or our understanding of gravity is wrong. The latter is highly unlikely since it is one of the most well-tested theories in physics. Thus there is likely a type of matter that weakly interacts light and regular matter that has been evading detection. We call this special type of matter ”dark matter” which is one of the open questions in modern physics.