Fraunhofer Diffraction

Corey Anderson

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1 Huygens-Fresnel Theory

Huygens developed the idea that light can be considered as a series of wavefronts where each new front creates the next wavefront by emitting a series of secondary disturbances - secondary wavelets - where the new wavefront is the envelop of these wavelets. Fresnel improved on this idea by adding that the secondary wavelets should interact by mutual interference. With this sophistication, Fresnel was able to use Huygens' idea to account for diffraction phenomena and hence physical optics.

2 Harmonic Spherical Waves

Waves created by a point source will radiate in all directions to produce a spherical wavefront. Mathematically, this is

$$\left(\frac{A}{r}\right)e^{ikr}, \tag{1}$$

where we have dropped time dependence because it is not relevant for diffraction problems.

3 Obliquity Factor

The issue with the previous construction is that wavelet is radiated in all directions equally. Fresnel solved this by introducing an obliquity factor

$$\kappa(\chi) = \frac{1 + \cos\chi}{2} \tag{2}$$

so that the wavelets travelling forward has the greatest intensity. Thus the secondary wavelets will have the form

$$\psi(r) = \left(\frac{A}{r}\right)e^{ikr}\kappa(\chi) \tag{3}$$

4 Far Field Condition

Fraunhofer diffraction requires the source to be place far from the aperture so that the light surface can be considered as having constant phase. This fact will be used in the derivation for the Fraunhofer condition.

5 Derivation of the Fraunhofer Diffraction Equation

We will use coordinates x and y for the aperture plane and coordinates ζ and η for the observation plane. Suppose we place an aperture function $f(x, y)$ in front of the observation plane. The aperture function will be 1 where it is transparant and 0 where it is opaque. If the aperture induces refraction, then the function will be complex with the exponent representing the phase change induced.

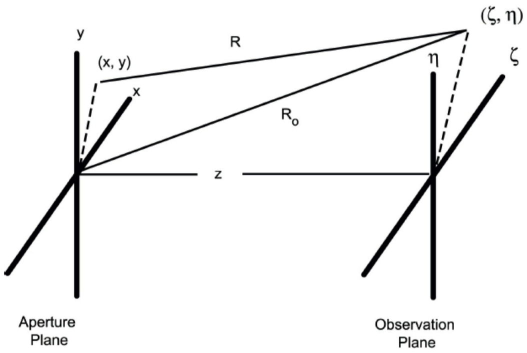

Figure 1: Aperture plane and observation planes

The aperture will act as a source for secondary wavelets of the form

$$\psi(x, y, \chi) = f(x, y)\frac{e^{ikR}}{R}\kappa(\chi). \tag{4}$$

The field arriving at the observation plane (ζ, η) can be computed by adding up all the secondary wavelet contributions arriving at (ζ, η). This results in the integral

$$\begin{aligned} \Phi(\zeta, \eta) &= \iint\limits_{aperture} \psi(x, y, \chi)\, dx\, dy & (5) \\ &= \iint\limits_{aperture} f(x, y)\frac{e^{ikR}}{R}\kappa(\chi)\, dx\, dy\,. & (6) \end{aligned}$$

Using or far-field condition, we can assume that $R \approx R_0$ and $\kappa(\chi) \approx$ const. However, we can not approximate R in the exponent because we would require the difference be extremely small. This means we have

$$\Phi(\zeta, \eta) = \frac{\kappa(\chi)}{R_0} \iint\limits_{aperture} f(x, y)e^{ikR}\, dx\, dy \tag{7}$$

To simplify Equation 7, we need to express R in terms of the coordinate variables. We can write

$$\begin{aligned} R^2 &= (\zeta - x)^2 + (\eta - y)^2 + z^2 & (8) \\ R_0^2 &= \zeta^2 + \eta^2 + z^2. & (9) \end{aligned}$$

Hence,

$$\begin{aligned} R^2 &= R_0^2 - 2(\zeta x - \eta y) + (x^2 + y^2) & (10) \\ &= R_0^2\left[1 + \frac{-2(\zeta x - \eta y) + (x^2 + y^2)}{R_0^2}\right]. & (11) \end{aligned}$$

R can be evaluated using the Taylor expansion

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - ... \tag{12}$$

Applying this, we get

$$R = R_0 - \frac{\zeta x - \eta y}{R_0} + \frac{x^2+y^2}{2R_0}. \tag{13}$$

The total diffracted wave is now

$$\Phi(\zeta,\eta) = \frac{\kappa(\chi)e^{ikR_0}}{R_0} \iint\limits_{aperture} f(x,y)e^{ik\alpha(x,y,\zeta,\eta)}\,dx\,dy \tag{14}$$

with

$$\alpha(x,y,\zeta,\eta) = -\frac{\zeta x - \eta y}{R_0} + \frac{x^2+y^2}{2R_0}. \tag{15}$$

The phase of the wave (exponent) is given by

$$\text{Phase} = k\alpha(x,y,\zeta,\eta) = 2\pi\left(-\frac{\zeta x - \eta y}{\lambda R_0} + \frac{x^2+y^2}{2\lambda R_0}\right). \tag{16}$$

Since we have assumed the aperture is small compared to the distance to the aperture,

$$\frac{x^2+y^2}{2\lambda R_0} << 1 \tag{17}$$

and so

$$\text{Phase} \approx -2\pi\frac{\zeta x - \eta y}{\lambda R_0} \tag{18}$$

and the diffracted field becomes

$$\Phi(\zeta,\eta) = \frac{\kappa(\chi)e^{ikR_0}}{R_0} \iint\limits_{aperture} f(x,y)e^{-2\pi i\frac{\zeta x - \eta y}{\lambda R_0}}\,dx\,dy\,. \tag{19}$$

To simplify even more, we use the change of variables

$$u = \frac{\sin\theta}{\lambda} \approx \frac{\zeta}{\lambda R_0} \qquad v = \frac{\sin\phi}{\lambda} \approx \frac{\eta}{\lambda R_0} \tag{20}$$

so that

$$\Phi(\zeta,\eta) = \frac{\kappa(\chi)e^{ikR_0}}{R_0} \iint\limits_{aperture} f(x,y)e^{-2\pi i(xu+yv)}\,dx\,dy \tag{21}$$

But because there is a limit on the size of the aperture (Equation 17), the aperture function $f(x,y)$ goes to zero when $x^2+y^2 > r_{max}^2$ so we can freely extend the integration bounds to $\pm\infty$.

$$\Phi(\zeta,\eta) = \frac{\kappa(\chi)e^{ikR_0}}{R_0} \int_0^\infty \int_0^\infty f(x,y)e^{-2\pi i(xu+yv)}\,dx\,dy \tag{22}$$

$$= \frac{\kappa(\chi)e^{ikR_0}}{R_0}\mathcal{F}\{f(x,y)\} \tag{23}$$

Equation 30 is just the diffraction field, to find the actual observed pattern, we need to find the intensity

$$I(u,v) = \Phi(\zeta,\eta)^*\Phi(\zeta,\eta) \tag{24}$$

$$= \left(\frac{\kappa(\chi)e^{ikR_0}}{R_0}\right)^2 \mathcal{F}^*\{f(x,y)\}\mathcal{F}\{f(x,y)\} \tag{25}$$

I hope you're amazed by this result, to find the diffraction pattern on the observation screen, all we need to do is take the Fourier transform of the aperture function! Note that this only holds when the condition in Equation 17 is met.