The Refractive Index

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Light travels slower in water than in air, and slightly slower in air than in a vacuum. This is quantified by the refractive index $n = c/v$, where c is the speed of light in a vacuum and v is the speed of light in the medium. In this report, we will quantitatively investigate why light slows down in mediums such as water and its implications.

1 Derivation

Suppose there exists a source at point S which is emitting EM radiation toward the point P through a thin plate made of some material. We will ignore the magnetic field component of the EM wave and consider only the electric field. The electric field from the source will interact with the electrons in the material and cause them to oscillate, which induces secondary electric fields since accelerating charges create EM radiation. The total field at P is the sum of all these contributions. Thus:

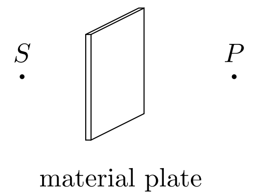

Figure 1: Electric waves passing through a thin plate of material

$$E_P = E_S + \sum_{\text{all electrons}} E_{\text{each electron}} \tag{1}$$

where E_P and E_S are the electric fields at P and S respectively, and $E_{\text{each electron}}$ is the electric field produced by one electron's oscillations. The field felt at P is modified by the material and it is modified in such a way that the field in the material seems to be moving at a different speed.

We will now derive the *form* of the electric field at P assuming there exists a refractive index n. For simplicity, I will assume the field generated by a charge does not interact with any other charges in the material. The source generates an electric field given by

$$E_S = E_0 e^{i\omega(t - z/c)}. \tag{2}$$

We know the field travels at a different speed v in the plate of thickness Δz. The time taken by light to pass through the plate is $n\Delta z/c$. Or, the time delay caused by the plate compared to a vacuum is $\Delta t = \Delta z/c - n\Delta z/c = (1-n)\Delta z/c$. Taking into account this time delay, Equation 2 becomes

$$E_{\text{after plate}} = E_0 e^{i\omega(t-(1-n)\Delta z/c - z/c)} \tag{3}$$
$$= E_0 e^{-i\omega(n-1)\Delta z/c} e^{i\omega(t-z/c)} \tag{4}$$

According to Equation 4, instead of the induced electric fields changing the amplitude of the original electric field, they instead add on a factor of $e^{-i\omega(n-1)\Delta z/c}$ which shifts the phase of the original wave. A block of some material can be thought of as many of these plates combined together. Each plate induces a phase shift and **it turns out that these phase shifts combined gives the illusion of light travelling at a different speed in the material**. Taking the second order power expansion, Equation 4 becomes

$$E_{\text{after plate}} = E_0 \left(1 - i\omega(n-1)\frac{\Delta z}{c}\right) e^{i\omega(t-z/c)} \tag{5}$$
$$= E_0 e^{i\omega(t-z/c)} - i\omega(n-1)\frac{\Delta z}{c} E_0 e^{i\omega(t-z/c)}. \tag{6}$$

The first term is from the source, and the second term is the field produced by the oscillating charges. Equation 6 will be used later.

We have now derived the **form** of the equation describing the consequences of the refractive index. We now need to do the most important bit - to show that a refractive index n shows up in the first place! We will assume that each electron in the material is fastened using springs and they are allowed to oscillate up and down in the x direction. The driving force will be E_S and the restoring force will be provided by the springs. Thus,

$$m\left(\frac{d^2x}{dt^2}+\omega_0 x\right)=eE_0e^{i\omega t}. \tag{7}$$

Note in Equation 7, $z=0$ since that's where the plate's position is defined to be at. Solving this second order differential equation gives the general solution

$$x=x_0e^{i\omega t}. \tag{8}$$

Substituting this back into Equation 7 returns

$$x_0=\frac{eE_0}{m(\omega_0^2-\omega^2)} \tag{9}$$

and thus

$$x=\frac{eE_0}{m(\omega_0^2-\omega^2)}e^{i\omega t}. \tag{10}$$

We now want to find the field produced by the sum of oscillating charges in the plate. Consider the setup in Figure 1.

The position of each charge at a given time can be represented as $x_0e^{i\omega t}$. The contribution to the field at point P is proportional to the acceleration of the corresponding charge at a retarded time, given by

$$-\omega_0^2x_0e^{-i\omega(t-r/c)}. \tag{11}$$

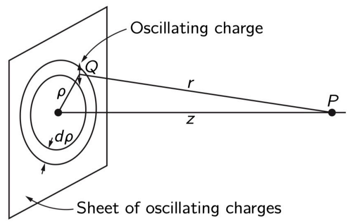

Figure 2: The field at P due to the sum of all the charges in the plate.

Now the equation for the actual electric field produced is far too tedious to prove so I will simply state it as

$$E'_P=\frac{-q}{4\pi\epsilon_0c^2r}a\left(t-\frac{r}{c}\right) \tag{12}$$

$$=\frac{e}{4\pi\epsilon_0c^2}\frac{\omega_0^2x_0e^{-i\omega(t-r/c)}}{r}. \tag{13}$$

This is only the field contribution for one charge, for all the charges we will set up an integral using η as the charge density. Thus, the total field at P is

$$E_P=\int\frac{e}{4\pi\epsilon_0c^2}\frac{\omega_0^2x_0e^{-i\omega(t-r/c)}}{\sqrt{\rho^2+z^2}}\eta\cdot 2\pi\rho\,d\rho \tag{14}$$

$$=-\frac{\eta e}{2\epsilon_0c}i\omega x_0e^{i\omega(t-z/c)} \tag{15}$$

We can now use Equation 9 to get

$$E_P=-\frac{\eta e}{2\epsilon_0c}i\omega\frac{eE_0}{m(\omega_0^2-\omega^2)}e^{i\omega(t-z/c)}. \tag{16}$$

As we expect, the oscillating charges in the plate produce a wave propagating to the right. Comparing Equation 16 with Equation 6, they will be equal when

$$(n-1)\Delta z=\frac{\eta e^2}{2\epsilon_0m(\omega_0^2-\omega^2)} \tag{17}$$

and noting that $\eta=N\Delta z$, where N is the number of electrons per unit volume:

$$n=1+\frac{Ne^2}{2\epsilon_0m(\omega_0^2-\omega^2)} \tag{18}$$

2 Consequences

In Section 1, we found that

$$n = \frac{c}{v} = 1 + \frac{Ne^2}{2\epsilon_0 m(\omega_0^2 - \omega^2)}. \tag{19}$$

We will consider several predictions based on this equation.

2.1 Dispersion

The index of refraction depends on the frequency ω of the light. The index increases for higher frequencies. This is known as *Dispersion* and it is the reason why blue light is bent more than red light through a prism.

2.2 Causality

A surprising consequence emerges if we imagine shining high frequency X-rays on matter, or radio waves on free electrons (free electrons have no restoring force so $\omega_0 = 0$). In these two cases, the factor $(\omega_0^2 - \omega^2)$ in Equation 19 becomes negative, which implies that $n = c/v < 1$ or $v > c$. Our whole framework appears to be wrong because information travelling faster than light violates causality! However, it turns out that what we have done is indeed correct because v is the phase velocity of light, i.e. the speed of each node of the wave. A node by itself is not a signal and cannot carry any information. To encode information into the wave, you've got to change the shape of the wave. To do this, the wave has to be constructed by multiple different frequency sinusoids. It turns out that the speed information travels at is not dependent on the index n alone, but upon the way n changes with the frequency. Lets now set out to demonstrate this.

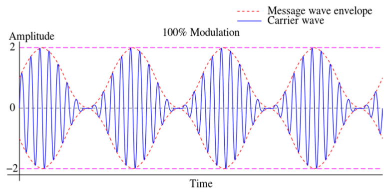

Figure 3: Multiple frequencies summed together to carry a signal

Phase Velocity:

Consider a sinusoidal wave $y = \cos(kx - \omega t)$. To find the speed of one crest on the wave we set the argument to zero, and differentiate with respect to t.

$$kx - \omega t = 0 \tag{20}$$

$$k\frac{dx}{dt} - \omega = 0 \tag{21}$$

$$v_{\text{phase}} = \frac{\omega}{k} \tag{22}$$

This is known as the phase velocity and it is the speed that individual peaks of the wave travels at.

Group Velocity:

Now lets construct a signal by adding two waves together:

$$e^{-i(k_1 z - \omega_1 t)} + e^{-i(k_2 z - \omega_2 t)} = e^{-i[(k_1+k_2)z-(\omega_1+\omega_2)t]/2}\left\{e^{i[(k_1-k_2)z-(\omega_1-\omega_2 t)]/2} + e^{-i[(k_1-k_2)z-(\omega_1-\omega_2 t)]/2}\right\}. \tag{23}$$

The factor in front of the brackets describes the wave itself (*Carrier Wave* in Figure 3), with speed $v_{\text{phase}} = \omega/k$. The expression inside the brackets describe the modulation wave (*Message Wave Envelope* in Figure 3). The modulation wave travels at a speed known as the group velocity:

$$v_{group} = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{d\omega}{dk} \tag{24}$$

If we made a signal; some kind of change in the wave that can be recognised if one listens to it, a modulation signal, then that modulation signal would travel at the group velocity. Therefore, **the group velocity is the speed that information propagates at**.

If we shine X-rays on a material, then $\omega_0^2 - \omega^2 \approx -\omega^2$, so Equation 19 becomes

$$n = 1 - \frac{Ne^2}{2\epsilon_0 m\omega^2}. \tag{25}$$

Using Equation 23, noting that $n = c/v = kc/\omega$ and rearranging,

$$k = \frac{\omega}{c} - \frac{a}{\omega c} \tag{26}$$

where $a = Ne^2/2\epsilon_0 m$. Differentiating Equation 26,

$$\frac{dk}{d\omega} = \frac{1}{c} + \frac{a}{\omega^2 c} \Longrightarrow \frac{d\omega}{dk} = \frac{c}{1 + a/\omega^2} = v_{\text{group}} \tag{27}$$

which is always less than or equal to the speed of light! So even though each crest of the wave may travel faster than c, the modulation signals (information carrying wave), never travels faster than c. Note that in a perfect vacuum, $N = 0$, so $a = 0$, and therefore $v_{\text{group}} = c$ which is the maximum speed that information can travel at.

2.3 Absorption

An improvement to Equation 19 is to consider the charges not as perfect oscillators, but damped oscillators. So in the denominator we change $(\omega_0^2 - \omega^2)$ to $(\omega_0^2 - \omega^2 + i\gamma\omega)$. Also the charges have multiple different resonant frequencies ω_k for $k = 0, 1, 2, ..., etc.$ If we include these effects, the more accurate model is described by

$$n = 1 + \frac{e^2}{2\epsilon_0 m} \sum_k \frac{N_k}{\omega_k^2 - \omega^2 + i\gamma_k\omega} \tag{28}$$

The term $i\gamma_k\omega$ in the denominator of Equation 20 means that the index of refraction is complex. That is, it's in the form $n = n_r - in_i$. To see what this means, we can put this back into Equation 4 to get

$$E_{\text{after plate}} = E_0 e^{-i\omega(n_r - in_i - 1)\Delta z/c} e^{i\omega(t - z/c)} \tag{29}$$

$$= e^{-\omega n_i \Delta z/c} e^{-i\omega(n_r - 1)\Delta z/c} E_0 e^{i\omega(t - z/c)}. \tag{30}$$

This is the same as Equation 4, except for the extra factor of $e^{-\omega n_i \Delta z/c}$ at the front. This is a quantity smaller than 1 which causes $E_{\text{after plate}}$ to decrease as the thickness of the plate Δz increases. This is expected since we have introduced a damping force in the oscillations which cause a loss of energy. The material is absorbing part of the wave. It is this effect that gives us the dark spectral lines of certain gasses. If the frequency is close to the resonant frequency of the atoms in the gas, $\omega \approx \omega_0$, then the index n is almost purely imaginary, $n_i >> n_r$. Therefore, the extra factor is very small and causes the dark absorption line in the spectrum ($E_{\text{after plate}} << 1$).