The Quantum Harmonic Oscillator

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In classical mechanics, oscillatory motion is governed by Hooke's law,

$$F = -kx = m\frac{d^2x}{dt^2} \tag{1}$$

with solution $x(t) = A\sin(\omega t) + B\cos(\omega t)$ where the angular frequency is $\omega = \sqrt{\frac{k}{m}}$.
The potential energy is $V(x) = \frac{1}{2}kx^2$. If we expand $V(x)$ around the local minimum x_0 we get

$$V(x) = V(x_0) + V'(x_0)(x - x_0) + \frac{1}{2}V''(x_0)(x - x_0)^2 + ..., \tag{2}$$

We can subtract the $V(x_0)$ since adding a constant doesn't change the force, also recognise $V(x_0) = 0$ and drop higher order terms, we get

$$\begin{aligned} V(x) &\approx \frac{1}{2}V''(x_0)(x - x_0)^2 &(3)\\ &\approx \frac{1}{2}m\omega^2(x - x_0)^2. &(4)\end{aligned}$$

The quantum problem requires us to solve the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2x^2\psi = E\psi. \tag{5}$$

which can be written as

$$\frac{1}{2m}\left[\hat{p}^2 + (m\omega x)^2\right]\psi = E\psi \tag{6}$$

We cannot factor the brackets using complex numbers because operators do not necessarily commute. This motivates the quantities

$$\hat{a}_\pm \equiv \frac{1}{\sqrt{2\hbar m\omega}}(\mp i\hat{p} + m\omega x) \tag{7}$$

The product $\hat{a}_-\hat{a}_+$ becomes

$$\begin{aligned} \hat{a}_-\hat{a}_+ &= \frac{1}{2\hbar m\omega}(i\hat{p} + m\omega x)(-i\hat{p} + m\omega x) &(8)\\ &= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega x)^2 - im\omega(x\hat{p} - \hat{p}x)\right] &(9)\\ &= \frac{1}{2\hbar m\omega}\left[\hat{p}^2 + (m\omega x)^2\right] - \frac{i}{2\hbar}[x, \hat{p}] &(10)\\ &= \frac{1}{\hbar\omega}\hat{H} + \frac{1}{2}. &(11)\end{aligned}$$

Similarly,

$$\hat{a}_+\hat{a}_- = \frac{1}{\hbar}\omega\hat{H} - \frac{1}{2}. \tag{12}$$

In particular, $[\hat{a}_-, \hat{a}_+] = 1$. Meanwhile, the Schrödinger equation takes the form

$$\hbar\omega\left(\hat{a}_\pm\hat{a}_\mp \pm \frac{1}{2}\right)\psi = E\psi \tag{13}$$

It can be shown that if ψ satisfies the Schrödinger equation with energy E, then $\hat{a}_+\psi$ satisfies the equation with energy $(E + \hbar\omega)$. Similarly, $\hat{a}_-\psi$ satisfies the equation with energy $(E - \hbar\omega)$.

There occurs a "lowest rung" such that $\hat{a}_-\psi_0 = 0$:

$$\frac{1}{\sqrt{2\hbar m\omega}}\left(\hbar\frac{d}{dx} + m\omega x\right)\psi_0 = 0 \tag{14}$$

and rearranging,

$$\frac{d\psi_0}{dx} = -\frac{m\omega}{\hbar}x\psi_0. \tag{15}$$

Whose solution becomes

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}. \tag{16}$$

To determine the energy of this state we plug it into the Schrödinger equation (13) to obtain

$$E_0 = \frac{1}{2}\hbar\omega. \tag{17}$$

Now that we have defined the ground state of the quantum oscillator, we simply apply the raising operator repeatedly to generate the excited states while increasing the energy by $\hbar\omega$ each step:

$$\psi_n(x) = A_n(\hat{a}_+)^n\psi_0(x), \quad \text{with } E_n = \left(n + \frac{1}{2}\right)\hbar\omega. \tag{18}$$

Now we need to find an expression for the coefficients so we are not required to normalise the wave function each time. We begin by noting that $\hat{a}_\pm$ is the hermitian conjugate of $\hat{a}_\mp$

$$\int_{-\infty}^{\infty} f * (\hat{a}_\pm g)\, dx = \int_{-\infty}^{\infty} (\hat{a}_\mp f) * g\, dx \tag{19}$$

so that

$$\int_{-\infty}^{\infty} (\hat{a}_\pm\psi_n)^*(\hat{a}_\pm\psi_n)\, dx = \int_{-\infty}^{\infty} (\hat{a}_\mp\hat{a}_\pm\psi_n) * \psi_n\, dx\,. \tag{20}$$

Using equation 13, we get

$$\hat{a}_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad \hat{a}_-\psi_n = \sqrt{n}\psi_{n-1}. \tag{21}$$

Thus

$$\psi_1 = \frac{1}{\sqrt{1!}}\hat{a}_+\psi_0, \qquad \psi_2 = \frac{1}{\sqrt{2}}\hat{a}_+\psi_1 = \frac{1}{\sqrt{2\cdot 1}}(\hat{a}_+)^2\psi_0, \qquad \psi_3 = \frac{1}{\sqrt{3}}\hat{a}_+\psi_2 = \frac{1}{\sqrt{3\cdot 2\cdot 1}}(\hat{a}_+)^3\psi_0$$

and we have

$$\psi_n = \frac{1}{\sqrt{n!}}(\hat{a}_+)^n\psi_0 \tag{22}$$

$$= \frac{1}{\sqrt{n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}(\hat{a}_+)^n e^{-\frac{m\omega}{2\hbar}x^2} \tag{23}$$

which is the general wave function of the quantum harmonic oscillator with energy level

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \tag{24}$$