A Paper On Black Holes That Doesn't Suck

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1 The Schwarzschild Metric

The Schwarzschild geometry defines the curvature of spacetime around a spherical mass. Its line element is defined by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \tag{1}$$

and the metric is therefore

$$g_{\alpha\beta} = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2}$$

where $ds^2 = g_{\alpha\beta}x^\alpha x^\beta$.

2 The Collapse of a Spherical Star

A star is born with the gravitational collapse of a cloud of hydrogen and helium. Compressional heating raises the temperature high enough so that nuclear fusion begins at the centre of the new star. The hydrogen is fused to produce helium and energy. The star reaches a state of dynamic equilibrium when the energy lost by radiation is balanced by the nuclear fusion reaction. The outward radiation pressure balances the gravitational However, there comes a point when the supply of hydrogen is exhausted. In this case, the star collapses further until the core temperature is enough to start the fusion of helium. This cycle of fusion continues up the periodic table until iron where the binding energy per nucleon is the highest. There is no more nuclear fuel in the star to burn at this point. The outward radiation pressure ceases and gravitational forces dominate causes the star to collapse into a black hole (assuming the gravitational force overcomes the fermi pressure).

3 Singularities in the Schwarzschild Metric

It is evident from Equation 1 that there exists two singularities at $r = 2M$ and $r = 0$. The singularity at $r = 2M$ turns out to be simply a coodinate singularity. To show this, we change coordinates systems into one which there is no singularity at $r = 2M$. There are multiple systems possible but a common one is the Eddington-Finkelstein coordinates which involves the transformation,

$$t = v - r - 2M \log\left|\frac{r}{2M} - 1\right| \tag{3}$$

where v is just a new introduced coordinate.

Substituting Equation 3 into Equation 1 gives,

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr + r^2\big(d\theta^2 + \sin^2\theta\, d\phi^2\big). \tag{4}$$

It is important to note that this is not a new geometry, it describes the same system as Equation 1 but with new coordinates labelling the points. Moreover, the singularity at $r = 2M$ has disappeared!

4 Light Cones of the Schwarzschild Geometry

To understand the Schwarzschild Geometry as a black hole we need to consider radial light rays. These are moving along world lines with $d\phi = d\theta = 0$ and $ds^2 = 0$. From Equation 4, this is

$$-\left(1 - \frac{2M}{r}\right) dv^2 + 2\, dv\, dr = 0. \tag{5}$$

One solution is

$$\boxed{v = \text{const} \qquad \text{(ingoing radial light rays)}} \tag{6}$$

We see these are ingoing light rays because from Equation 3, when t increases, r must decrease to keep v constant. Another solution is

$$-\left(1 - \frac{2M}{r}\right) dv + 2\, dr = 0, \tag{7}$$

which can be integrated to give

$$v - 2\left(r + 2M \log\left|\frac{r}{2M} - 1\right|\right) = \text{const.} \qquad \begin{cases} \text{Outgoing } r > 2M \\ \text{Ingoing } r < 2M \end{cases} \tag{8}$$

When the photon is far from the black hole, one radial ray travels inward and one outward. When it is close to the black hole $r < 2M$, both radial rays fall inward. At $r = 2M$, one ray falls inward but one remains stationary. It is obvious that something significant occurs at $r = 2M$. This point is known as the event horizon of a black hole. If one proceeds within this point, it is impossible to escape.

Figure 1 is a spacetime diagram showing the radial light rays in Eddington-Finkelstein's coordinates. Null lines of constant v have been plotted at 45° angle by using $\tilde{t} = v - r$ on the y-axis. The radial light rays at $r = 2M$ are indicated by a solid black line. Light cones at a few intersections are indicated. Note that the light rays tip more as they approach the singularity $r = 0$. For $r < 2M$, the future of each light cone points toward the singularity which means that the future of an observer falling inside the event horizon become the singularity itself. The event horizon thus divides spacetime into two regions, one where light can escape to infinity, and one where the gravitational force is so strong that not even light can escape.

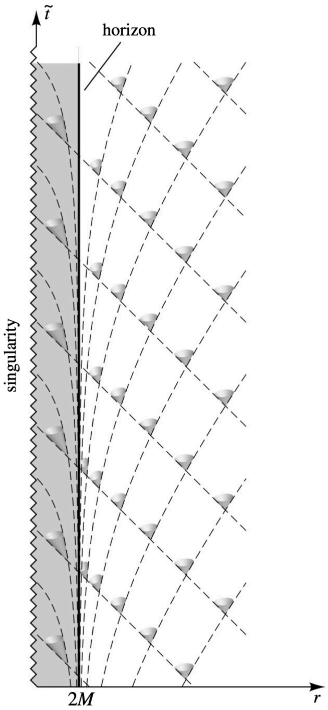

Figure 1: The radial light rays of Schwarzschild Geometry