Special Relativity

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Contents

1 Special Relativistic Kinematics

The world line is the path traced out by a particle moving through spacetime. To make the world line 4 dimensional, we will specify 4 coordinates of the particle's position at x^μ . Many parameters are possible for the position, but a natural choice is the proper time τ. Thus, a world line is described by

$$x^\mu = x^\mu(\tau). \tag{1}$$

The proper time is measured by a clock carried along the world line.

1.1 4-Position

Since we have 4 coordinates - three spatial, one temporal, we make the following definition of 4-position:

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{2}$$

1.2 4-Velocity

The four-velocity has components u^μ which are the derivatives of the position along the world line with respect to the proper time τ:

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma \frac{dx^\mu}{dt} = \gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix}. \tag{3}$$

Lets now find the length of this vector:

$$u^\mu u_\nu = u^\mu \eta_{\mu\nu} u^\nu = \gamma^2 (c^2 - v^2) = c^2. \tag{4}$$

1.3 4-Momentum

We can define the components of the 4-momentum as:

$$p^\mu = m u^\mu = m\gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix} \tag{5}$$

Finding the length of this vector:

$$p^\mu p_\nu = p^\mu \eta_{\mu\nu} p^\nu = m^2 \gamma^2 (c^2 - v^2) = m^2 c^2 \tag{6}$$

1.4 Energy-Momentum

Note that the 3-momentum in the previous section can be written as

$$\vec{p} = m\gamma\vec{v} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{7}$$

and is conserved in all inertial frames.

Lets now consider the kinetic energy as the amount of work required to accelerate a particle of mass m to a velocity v:

$$
\begin{aligned}
E_K = W &= \int_{x_i}^{x_f} F dx && (8)\\
&= \int_{x_i}^{x_f} \frac{dp}{dt} dx && (9)\\
&= \int_0^v \frac{dp}{dt}\frac{dx}{dp} dp && (10)\\
&= \int_0^v \frac{dx}{dt} dp && (11)\\
&= \int_0^v u dp && (12)\\
&= \left[up \right|_0^v - \int_0^v p du && (13)\\
&= \left[\frac{mc^2}{\sqrt{1-(\frac{u}{c})^2}} \right|_0^v && (14)\\
&= mc^2 \left(\frac{1}{\sqrt{1-(\frac{v}{c})^2}} - 1 \right). && (15)
\end{aligned}
$$

Thus we arrive at the relativistic kinetic energy,

$$
\begin{aligned}
E_K &= \gamma mc^2 - mc^2 && (16)\\
\implies \gamma mc^2 &= E_K + mc^2 = E. && (17)
\end{aligned}
$$

The term γmc^2 is the total energy of the system and is the sum of the kinetic energy and the term mc^2 which we'll interpret later on. We can now adjust the 4-momentum to be

$$
\begin{aligned}
p^\mu &= m\gamma \begin{pmatrix} c \\ \vec{v} \end{pmatrix} && (18)\\
&= \gamma \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}. && (19)
\end{aligned}
$$

We can now write

$$p^\mu p_\nu = \frac{E^2}{c^2} - p^2 = m^2c^2. \tag{20}$$

So

$$\boxed{E^2 = p^2c^2 + m^2c^4} \tag{21}$$

which is the energy-momentum relation. For a stationary particle, $p = 0$ so we have arguably the most famous formula in physics which is $E = mc^2$. This is the equation that gives the energy of the rest mass. Looking back at Equation 17, we see that the total energy is actually the sum of the particle's kinetic energy plus its rest mass.